Special factors and images of Arnoux-Rauzy words

Michelangelo Bucci     Alessandro De Luca

Dipartimento di Matematica e Applicazioni “R. Caccioppoli”
Università degli Studi di Napoli Federico II

Journées Montoises d’Informatique Théorique, Mons 30/08/2008
Our Goal

We will prove the following:

**Main Theorem**

Let $N \in \mathbb{N}$ and $s \in A^\omega$ be such that for any $k \geq N$:

1. $s$ has at most one left special factor of length $k$ (which must be a prefix of $s$),
2. $s$ has at most one right special factor of length $k$.

Then there exists $B \subseteq A$ and a standard Arnoux-Rauzy word $t \in B^\omega$ such that $s$ is a morphic image of $t$, under an injective morphism.
Our Goal

We will prove the following:

**Main Theorem**

Let $N \in \mathbb{N}$ and $s \in A^\omega$ be such that for any $k \geq N$:

1. $s$ has at most one left special factor of length $k$ (which must be a prefix of $s$),
2. $s$ has at most one right special factor of length $k$.

Then there exists $B \subseteq A$ and a standard Arnoux-Rauzy word $t \in B^\omega$ such that $s$ is a morphic image of $t$, under an injective morphism.
Our Goal

We will prove the following:

Main Theorem

Let \( N \in \mathbb{N} \) and \( s \in A^\omega \) be such that for any \( k \geq N \):

1. \( s \) has at most one left special factor of length \( k \) (which must be a prefix of \( s \)),
2. \( s \) has at most one right special factor of length \( k \).

Then there exists \( B \subseteq A \) and a standard Arnoux-Rauzy word \( t \in B^\omega \) such that \( s \) is a morphic image of \( t \), under an injective morphism.
Outline

1. Factor complexity and special factors
   - Minimal complexity: Sturmian words
   - Arnoux-Rauzy words and generalizations

2. Sketching the proof
   - Return words and recurrence
   - The proof, properly

3. Conclusions
1 Factor complexity and special factors
   - Minimal complexity: Sturmian words
   - Arnoux-Rauzy words and generalizations

2 Sketching the proof
   - Return words and recurrence
   - The proof, properly

3 Conclusions
Definitions

- Let \( w \) be a finite or infinite word. The factor complexity of \( w \) is the function \( c_w : \mathbb{N} \rightarrow \mathbb{N} \) counting the factors of \( w \) of each length:

\[
c_w(n) = \text{card}(A^n \cap \text{Fact } w).
\]

- The right (resp. left) degree of \( u \in \text{Fact } w \) is the number of different letters \( a \) such that \( ua \) (resp. \( au \)) is a factor of \( w \).

- A factor is right (resp. left) special if its right (resp. left) degree is greater than 1.
The Basics

Definitions

Let \( w \) be a finite or infinite word. The factor complexity of \( w \) is the function \( c_w : \mathbb{N} \rightarrow \mathbb{N} \) counting the factors of \( w \) of each length:

\[
c_w(n) = \text{card}(A^n \cap \text{Fact } w).
\]

The right (resp. left) degree of \( u \in \text{Fact } w \) is the number of different letters \( a \) such that \( ua \) (resp. \( au \)) is a factor of \( w \).

A factor is right (resp. left) special if its right (resp. left) degree is greater than 1.
Definitions

Let $w$ be a finite or infinite word. The factor complexity of $w$ is the function $c_w : \mathbb{N} \rightarrow \mathbb{N}$ counting the factors of $w$ of each length:

$$c_w(n) = \text{card}(A^n \cap \text{Fact } w).$$

The right (resp. left) degree of $u \in \text{Fact } w$ is the number of different letters $a$ such that $ua$ (resp. $au$) is a factor of $w$.

A factor is right (resp. left) special if its right (resp. left) degree is greater than 1.
Morse-Hedlund Theorem

(Well-known variant:)

**Theorem**

An infinite word $s$ is ultimately periodic if and only if

$$c_s(n) = c_s(n + 1)$$

for some $n \geq 0$, that is, iff $s$ has no right special factors of length $n$.

**Corollary**

- $s$ is ultimately periodic $\iff c_s(n) \leq n$ for some $n$.
- $s$ is (purely) periodic $\iff s$ has no left special factors of some length $n$. 
Morse-Hedlund Theorem

(Well-known variant:)

**Theorem**

An infinite word $s$ is ultimately periodic if and only if

$$c_s(n) = c_s(n + 1)$$

for some $n \geq 0$, that is, iff $s$ has no right special factors of length $n$.

**Corollary**

- $s$ is ultimately periodic $\iff c_s(n) \leq n$ for some $n$.
- $s$ is (purely) periodic $\iff s$ has no left special factors of some length $n$. 
Morse-Hedlund Theorem

(Well-known variant:)

**Theorem**

An infinite word $s$ is ultimately periodic if and only if

$$c_s(n) = c_s(n + 1)$$

for some $n \geq 0$, that is, iff $s$ has no right special factors of length $n$.

**Corollary**

- $s$ is ultimately periodic $\iff c_s(n) \leq n$ for some $n$.
- $s$ is (purely) periodic $\iff s$ has no left special factors of some length $n$. 
Sturmian Words

Definition

A word $s$ is **Sturmian** if $c_s(n) = n + 1$ for all $n \geq 0$.

Example

The Fibonacci word $f = abababaababaababaababaababaababaababaababa\cdots$ is Sturmian; we have e.g. $c_f(5) = 6$, as

$$A^5 \cap \text{Fact } f = \{aabaa, aabab, abaab, ababa, baaba, babaa\}.$$

The unique right special factor of length 4 in $f$ is $aabab$. 
Sturmian Words

Definition

A word \( s \) is **Sturmian** if \( c_s(n) = n + 1 \) for all \( n \geq 0 \).

Example

The Fibonacci word \( f = abababaabaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaababaabala

A^5 \cap \text{Fact } f = \{aabaa, aabab, abaab, ababa, baaba, babaa\}.

The unique right special factor of length 4 in \( f \) is \( aaba \).
Sturmian Words

Definition

A word $s$ is Sturmian if $c_s(n) = n + 1$ for all $n \geq 0$.

Example

The Fibonacci word $f = abababaababaababaababaababaababaababaababaababaababaababa \cdots$ is Sturmian; we have e.g. $c_f(5) = 6$, as

$$A^5 \cap \text{Fact } f = \{aabaa, aabab, abaab, ababa, baaba, babaa\}.$$ 

The unique right special factor of length 4 in $f$ is $aaba$. 
Arnoux-Rauzy Words

Definition

A recurrent word $s \in A^\omega$ is Arnoux-Rauzy (or strict episturmian) if it has exactly one left special factor and one right special factor (of degree $\text{card } A$) of each length $n \geq 0$.

($a^\omega = aaaaa \cdots$ is AR over $\{a\}$.)

Example

The Tribonacci word

$$\tau = abacabaabacababacabaabacabaca \cdots$$

is an Arnoux-Rauzy word.
Arnoux-Rauzy Words

Definition

A recurrent word $s \in A^\omega$ is Arnoux-Rauzy (or strict episturmian) if it has exactly one left special factor and one right special factor (of degree $\text{card } A$) of each length $n \geq 0$.

($a^\omega = aaaa \cdots$ is AR over $\{a\}$.)

Example

The Tribonacci word

$$\tau = abacabaabacababacababaabacabaca \cdots$$

is an Arnoux-Rauzy word.
The Reversal Operator

The set of factors of an Arnoux-Rauzy word is closed under the reversal operator (~), which maps any word \( w = a_1 a_2 \cdots a_n \) (with \( a_i \in A \) for \( 1 \leq i \leq n \)) to its mirror image \( \tilde{w} = a_n \cdots a_2 a_1 \).

This leads to a generalization:

**Definition (Droubay, Justin, Pirillo)**

An infinite word \( s \) is episturmian if it has at most one right special factor of each length, and \( \text{Fact } s \) is closed under reversal.
The Reversal Operator

The set of factors of an Arnoux-Rauzy word is closed under the reversal operator (\(\sim\)), which maps any word \(w = a_1 a_2 \cdots a_n\) (with \(a_i \in A\) for \(1 \leq i \leq n\)) to its mirror image \(\tilde{w} = a_n \cdots a_2 a_1\).

This leads to a generalization:

**Definition (Droubay, Justin, Pirillo)**

An infinite word \(s\) is **episturmian** if it has at most one right special factor of each length, and \(\text{Fact } s\) is closed under reversal.
Further Generalizations

The reversal operator is a particular involutory antimorphism of $A^*$, as $\bar{u}v = \bar{v}\bar{u}$ and $\bar{u} = u$ for all $u, v \in A^*$.

Let $\theta$ be any involutory antimorphism of $A^*$.

Definitions (Bucci, de Luca, De Luca, Zamboni)

An infinite word $s$ is $\theta$-episturmian if it has at most one right special factor of each length, and Fact $s$ is closed under $\theta$.

An infinite word $s$ is a $\theta$-word with seed if it has at most one right special factor of each length greater than some $N \in \mathbb{N}$, and Fact $s$ is closed under $\theta$. 
Further Generalizations

The reversal operator is a particular involutory antimorphism of $A^*$, as $\overline{uv} = \overline{v}\overline{u}$ and $\overline{u} = u$ for all $u, v \in A^*$.

Let $\theta$ be any involutory antimorphism of $A^*$.

Definitions (Bucci, de Luca, De Luca, Zamboni)

An infinite word $s$ is $\theta$-episturmian if it has at most one right special factor of each length, and $\text{Fact } s$ is closed under $\theta$.

An infinite word $s$ is a $\theta$-word with seed if it has at most one right special factor of each length greater than some $N \in \mathbb{N}$, and $\text{Fact } s$ is closed under $\theta$. 
Further Generalizations

The reversal operator is a particular involutory antimorphism of $A^*$, as $\tilde{uv} = \tilde{v} \tilde{u}$ and $\tilde{u} = u$ for all $u, v \in A^*$.

Let $\vartheta$ be any involutory antimorphism of $A^*$.

**Definitions (Bucci, de Luca, De Luca, Zamboni)**

An infinite word $s$ is $\vartheta$-episturmian if it has at most one right special factor of each length, and $\text{Fact } s$ is closed under $\vartheta$.

An infinite word $s$ is a $\vartheta$-word with seed if it has at most one right special factor of each length greater than some $N \in \mathbb{N}$, and $\text{Fact } s$ is closed under $\vartheta$. 
The Standard Case

All these families of words have standard elements, in which the unique left special factors are prefixes.

Standard words are good representatives, that is:

**Proposition**

An infinite word is Sturmian (or Arnoux-Rauzy, $\delta$-episturmian, etc.) if and only if there exists a standard Sturmian (Arnoux-Rauzy, ...) word having the same set of factors.
The Standard Case

All these families of words have standard elements, in which the unique left special factors are prefixes.

Standard words are good representatives, that is:

**Proposition**

An infinite word is Sturmian (or Arnoux-Rauzy, $\vartheta$-episturmian, etc.) if and only if there exists a standard Sturmian (Arnoux-Rauzy, \ldots) word having the same set of factors.
1. Factor complexity and special factors
   - Minimal complexity: Sturmian words
   - Arnoux-Rauzy words and generalizations

2. Sketching the proof
   - Return words and recurrence
   - The proof, properly

3. Conclusions
Our Goal Again

The standard words of all families considered so far are morphic images of standard Arnoux-Rauzy words.

Thus our

Main Theorem

Let $N \in \mathbb{N}$ and $s \in A^\omega$ be such that for any $k \geq N$:

1. $s$ has at most one left special factor of length $k$ (which must be a prefix of $s$),
2. $s$ has at most one right special factor of length $k$.

Then there exists $B \subseteq A$ and a standard Arnoux-Rauzy word $t \in B^\omega$ such that $s$ is a morphic image of $t$, under an injective morphism.

establishes the same connection with Arnoux-Rauzy words, for a wider class of sequences in which no “closure” assumption is made.
Our Goal Again

The standard words of all families considered so far are morphic images of standard Arnoux-Rauzy words.

Thus our

Main Theorem

Let $N \in \mathbb{N}$ and $s \in A^\omega$ be such that for any $k \geq N$:

1. $s$ has at most one left special factor of length $k$ (which must be a prefix of $s$),

2. $s$ has at most one right special factor of length $k$.

Then there exists $B \subseteq A$ and a standard Arnoux-Rauzy word $t \in B^\omega$ such that $s$ is a morphic image of $t$, under an injective morphism.

establishes the same connection with Arnoux-Rauzy words, for a wider class of sequences in which no “closure” assumption is made.
Return Words

Let $w$ be a factor of an infinite word $s$.

**Definition**

If $z = uw = wv \in \text{Fact } s$ contains exactly two occurrences of $w$, then $u$ is a return word to $w$ in $s$.

**Example**

Define a morphism $\mu$ by $\mu(a) = ab$, $\mu(b) = aac$, and let

$$s = \mu(f) = abaaacababaacababaacababaacababaacababa \cdots .$$

Then $ab$ and $abaac$ are return words to $w = aba$ in $s$. Note that $s$ satisfies all hypotheses of our theorem, but Fact $s$ is not closed under any involutory antimorphism.
Return Words

Let $w$ be a factor of an infinite word $s$.

**Definition**

If $z = uw = wv \in \text{Fact } s$ contains exactly two occurrences of $w$, then $u$ is a *return word* to $w$ in $s$.

**Example**

Define a morphism $\mu$ by $\mu(a) = ab$, $\mu(b) = aac$, and let

$s = \mu(f) = abaac ab \underline{aba} ac abaac \underline{aba} baacabaabaacababa \cdots$.

Then $ab$ and $abaac$ are return words to $w = aba$ in $s$.

Note that $s$ satisfies all hypotheses of our theorem, but $\text{Fact } s$ is not closed under any involutory antimorphism.
Return Words

Let $w$ be a factor of an infinite word $s$.

**Definition**

If $z = uw = vw \in \text{Fact } s$ contains exactly two occurrences of $w$, then $u$ is a return word to $w$ in $s$.

**Example**

Define a morphism $\mu$ by $\mu(a) = ab$, $\mu(b) = aac$, and let

$$s = \mu(f) = ab aac \underline{ab} a ba a c \underline{abaac} \underline{aba} ba a cacaba ba a cacababa a a c a c a baba \cdots .$$

Then $ab$ and $abaac$ are return words to $w = aba$ in $s$.

Note that $s$ satisfies all hypotheses of our theorem, but $\text{Fact } s$ is not closed under any involutory antimorphism.
Return Words

Let \( w \) be a factor of an infinite word \( s \).

**Definition**

If \( z = uw = vw \in \text{Fact} \ s \) contains exactly two occurrences of \( w \), then \( u \) is a return word to \( w \) in \( s \).

**Example**

Define a morphism \( \mu \) by \( \mu(a) = ab, \mu(b) = aac \), and let

\[
s = \mu(f) = abaac ab \underline{aba} ac ab aac ab baacababaacababaacababa \cdots .
\]

Then \( ab \) and \( abaac \) are return words to \( w = aba \) in \( s \).

Note that \( s \) satisfies all hypotheses of our theorem, but \( \text{Fact} \ s \) is not closed under any involutory antimorphism.
Two More Lemmas...

Lemma

Let $s \in A^\omega$ be such that any left special factor of sufficient length is a prefix of $s$. Then $s$ is recurrent.

Lemma

Let $s$ be recurrent and aperiodic. Then every factor $\omega$ of $s$ is contained in some bispecial (i.e., left and right special) factor of $s$. In particular, $s$ has infinitely many bispecial factors.
Two More Lemmas...

**Lemma**

Let \( s \in A^\omega \) be such that any left special factor of sufficient length is a prefix of \( s \). Then \( s \) is recurrent.

**Lemma**

Let \( s \) be recurrent and aperiodic. Then every factor \( w \) of \( s \) is contained in some bispecial (i.e., left and right special) factor of \( s \).

In particular, \( s \) has infinitely many bispecial factors.
Two More Lemmas...

Lemma

Let $s \in A^\omega$ be such that any left special factor of sufficient length is a prefix of $s$. Then $s$ is recurrent.

Lemma

Let $s$ be recurrent and aperiodic. Then every factor $w$ of $s$ is contained in some \textit{bispecial} \textit{(i.e., left and right special)} factor of $s$. In particular, $s$ has infinitely many bispecial factors.
Proof of the Theorem (sketch)

- If $s$ has no left special factors of some length $n$, then it is periodic and hence a morphic image of $x^\omega$ for any $x \in A$.
- Then suppose that $s$ has exactly one left special factor from length $N$ on. This implies that $s$ is aperiodic and recurrent.
- Thus we can consider a sequence $(W_i)_{i \geq 0}$ of bispecial factors of $s$, ordered by increasing length starting from $|W_0| \geq N$.
- By uniqueness, $W_i \in \text{Pref } W_{i+1} \cap \text{Suff } W_{i+1}$ for all $i \geq 0$, so that the sequence of corresponding degrees is non-increasing and hence eventually constant.
Proof of the Theorem (sketch)

- If $s$ has no left special factors of some length $n$, then it is periodic and hence a morphic image of $x^\omega$ for any $x \in A$.
- Then suppose that $s$ has exactly one left special factor from length $N$ on. This implies that $s$ is aperiodic and recurrent.
- Thus we can consider a sequence $(W_i)_{i \geq 0}$ of bispecial factors of $s$, ordered by increasing length starting from $|W_0| \geq N$.
- By uniqueness, $W_i \in \text{Pref } W_{i+1} \cap \text{Suff } W_{i+1}$ for all $i \geq 0$, so that the sequence of corresponding degrees is non-increasing and hence eventually constant.
Proof of the Theorem (sketch)

- If $s$ has no left special factors of some length $n$, then it is periodic and hence a morphic image of $x^\omega$ for any $x \in A$.

- Then suppose that $s$ has exactly one left special factor from length $N$ on. This implies that $s$ is aperiodic and recurrent.

- Thus we can consider a sequence $(W_i)_{i \geq 0}$ of bispecial factors of $s$, ordered by increasing length starting from $|W_0| \geq N$.

- By uniqueness, $W_i \in \text{Pref } W_{i+1} \cap \text{Suff } W_{i+1}$ for all $i \geq 0$, so that the sequence of corresponding degrees is non-increasing and hence eventually constant.
Proof of the Theorem (sketch)

- If $s$ has no left special factors of some length $n$, then it is periodic and hence a morphic image of $x^\omega$ for any $x \in A$.
- Then suppose that $s$ has exactly one left special factor from length $N$ on. This implies that $s$ is aperiodic and recurrent.
- Thus we can consider a sequence $(W_i)_{i \geq 0}$ of bispecial factors of $s$, ordered by increasing length starting from $|W_0| \geq N$.
- By uniqueness, $W_i \in \text{Pref } W_{i+1} \cap \text{Suff } W_{i+1}$ for all $i \geq 0$, so that the sequence of corresponding degrees is non-increasing and hence eventually constant.
Proof of the Theorem (cont.)

- Let $w = W_k$ be the first element of the sequence having the smallest degree, and let $B = \{ x \in A \mid xw \in \text{Fact } s \}$.
- By hypothesis and by definition of return word, it turns out that any two distinct return words to $w$ in $s$ have distinct final letters.
- Thus we can define a morphism $\varphi : B^* \to A^*$ mapping each letter to the unique return word ending with such letter.
- This way, we have $s = \varphi(t)$ for some $t \in B^\omega$ (which is a derivated word of $s$).
Let \( w = W_k \) be the first element of the sequence having the smallest degree, and let \( B = \{ x \in A \mid xw \in \text{Fact } s \} \).

By hypothesis and by definition of return word, it turns out that any two distinct return words to \( w \) in \( s \) have distinct final letters.

Thus we can define a morphism \( \varphi : B^* \rightarrow A^* \) mapping each letter to the unique return word ending with such letter.

This way, we have \( s = \varphi(t) \) for some \( t \in B^\omega \) (which is a \textit{derivated word} of \( s \)).
Let $w = W_k$ be the first element of the sequence having the smallest degree, and let $B = \{ x \in A \mid xw \in \text{Fact } s \}$. By hypothesis and by definition of return word, it turns out that any two distinct return words to $w$ in $s$ have distinct final letters. Thus we can define a morphism $\varphi : B^* \rightarrow A^*$ mapping each letter to the unique return word ending with such letter. This way, we have $s = \varphi(t)$ for some $t \in B^\omega$ (which is a derived word of $s$).
Proof of the Theorem (cont.)

- Let \( w = W_k \) be the first element of the sequence having the smallest degree, and let \( B = \{ x \in A \mid xw \in \text{Fact } s \} \).

- By hypothesis and by definition of return word, it turns out that any two distinct return words to \( w \) in \( s \) have distinct final letters.

- Thus we can define a morphism \( \varphi : B^* \to A^* \) mapping each letter to the unique return word ending with such letter.

- This way, we have \( s = \varphi(t) \) for some \( t \in B^\omega \) (which is a derived word of \( s \)).
Proof (conclusion)

- By the definition of return word, we have

**Lemma**

- \( z \in \text{Fact } t \iff \varphi(z)w \in \text{Fact } s \),
- \( z \in \text{Pref } t \iff \varphi(z)w \in \text{Pref } s \).

- From this and the hypotheses on \( s \), we obtain that \( t \) has exactly one right special factor of each length, all of degree \( \text{card } B \), and that left special factors of \( t \) are prefixes of it.

- That is, \( t \) is a standard Arnoux-Rauzy word over \( B \). □
Proof (conclusion)

- By the definition of return word, we have

**Lemma**

- $z \in \text{Fact } t \iff \varphi(z)w \in \text{Fact } s,$
- $z \in \text{Pref } t \iff \varphi(z)w \in \text{Pref } s.$

- From this and the hypotheses on $s$, we obtain that $t$ has exactly one right special factor of each length, all of degree $\text{card } B$, and that left special factors of $t$ are prefixes of it.

- That is, $t$ is a standard Arnoux-Rauzy word over $B$. □
Proof (conclusion)

By the definition of return word, we have

**Lemma**

- $z \in \text{Fact } t \iff \varphi(z)w \in \text{Fact } s$,
- $z \in \text{Pref } t \iff \varphi(z)w \in \text{Pref } s$.

From this and the hypotheses on $s$, we obtain that $t$ has exactly one right special factor of each length, all of degree $\text{card } B$, and that left special factors of $t$ are prefixes of it.

That is, $t$ is a standard Arnoux-Rauzy word over $B$. □
In Other Words

Example

\[ s = abaa cababaa cabaacababaa cababaa cababaa cabaacababaa c \cdots . \]

- \( N = 0. \)
- Bispecial factors: \( \varepsilon, a, aba, abaaababa, abaaabababaababa, \ldots \)
- \( w = aba, B = \{ b, c \}. \)
In Other Words

Example

\[ s = abaa cabaa cabaa cabaa cabaa cabaa cabaa cabaa cabaa cabaa cabaa \cdots . \]

- \( N = 0. \)
- Bispecial factors: \( \epsilon, a, aba, abaacaba, abaacababaacaba, \ldots \)
- \( w = aba, B = \{b, c\}. \)
In Other Words

Example

\[ s = abaa c a b a b a a c a b a a a c a b a a a c a b a a c a b a a a c a b a a a c a b a a a c a b a a a c \cdots. \]

- \( N = 0. \)
- Bispecial factors: \( \varepsilon, a, aba, abaacaba, abaacababaacaba, \ldots \)
- \( w = aba, B = \{b, c\}. \)
In Other Words

Example

\[ s = abaa c|a b|abaa c|abaa c|a b|abaa c|a b|abaa c|abaa c|a b|abaa c \cdots \]

- \( N = 0 \).
- Bispecial factors: \( \epsilon, a, aba, abaacaba, abaacababaacaba, \ldots \)
- \( w = aba, B = \{ b, c \} \).
Example

\[ t = \ c \ b \ c \ c \ b \ c \ b \ c \ c \ b \ c \cdots . \]

- \( N = 0 \).
- Bispecial factors: \( \varepsilon, a, aba, abaacaba, abaacababaacaba, \ldots \)
- \( w = aba, B = \{b, c\} \).
Factor complexity and special factors
- Minimal complexity: Sturmian words
- Arnoux-Rauzy words and generalizations

Sketching the proof
- Return words and recurrence
- The proof, properly

Conclusions
Summary

1. We have considered (standard) Arnoux-Rauzy words and several generalizations,
2. introduced a further extension, based on a property of special factors,
3. and finally shown that all words in such class are morphic images of standard Arnoux-Rauzy words.

Open questions:
- Full analysis of the general (non-standard) case,
- Converse problem: does every morphic image of a standard Arnoux-Rauzy word satisfy the hypotheses of our theorem? If not, which ones do?
Summary

1. We have considered (standard) Arnoux-Rauzy words and several generalizations,
2. introduced a further extension, based on a property of special factors,
3. and finally shown that all words in such class are morphic images of standard Arnoux-Rauzy words.

Open questions:
- Full analysis of the general (non-standard) case,
- Converse problem: does every morphic image of a standard Arnoux-Rauzy word satisfy the hypotheses of our theorem? If not, which ones do?
Summary

1. We have considered (standard) Arnoux-Rauzy words and several generalizations,
2. introduced a further extension, based on a property of special factors,
3. and finally shown that all words in such class are morphic images of standard Arnoux-Rauzy words.

Open questions:

- Full analysis of the general (non-standard) case,
- Converse problem: does every morphic image of a standard Arnoux-Rauzy word satisfy the hypotheses of our theorem? If not, which ones do?
Summary

1. We have considered (standard) Arnoux-Rauzy words and several generalizations,
2. introduced a further extension, based on a property of special factors,
3. and finally shown that all words in such class are morphic images of standard Arnoux-Rauzy words.

Open questions:
- Full analysis of the general (non-standard) case,
- Converse problem: does every morphic image of a standard Arnoux-Rauzy word satisfy the hypotheses of our theorem? If not, which ones do?
Summary

1. We have considered (standard) Arnoux-Rauzy words and several generalizations,
2. introduced a further extension, based on a property of special factors,
3. and finally shown that all words in such class are morphic images of standard Arnoux-Rauzy words.

Open questions:
- Full analysis of the general (non-standard) case,
- Converse problem: does every morphic image of a standard Arnoux-Rauzy word satisfy the hypotheses of our theorem? If not, which ones do?